Multinomial:

$$n^{-k/2} \sum_{|\alpha|=k} \binom{k}{\alpha_1, \dots, \alpha_n} \prod_{i=1}^{n} \mathbb{E}(X_i^{\alpha_i})$$

$$\alpha_1 + \cdots + \alpha_n = k$$

$$(\alpha_i \geq 0)$$

Helps to consider instead the multinomial α to have only nonzero terms, & let the number of those nonzero terms vary:

$$n^{-k/2} \sum_{\alpha} \sum_{1 \leq i_1 < \cdots < i_{\#\text{terms}} \leq n} \binom{k}{\alpha_1, \dots, \alpha_{\#\text{terms}}} \prod_{j=1}^{\#\text{terms}} \mathbb{E}(X_{i_j}^{\alpha_j})$$

$$\alpha_1 + \cdots + \alpha_{\#\text{terms}} = k$$

$$(\alpha_i > 0)$$

These α are "strong compositions" of k.
Strong means no zeros allowed;
compositions (not partitions) because
order of terms matters.

(The order of the terms α_i
doesn't change $\binom{k}{\alpha_1, \dots, \alpha_{\#\text{terms}}}$, but
to count the terms properly
we're taking $x_{i_1}, \dots, x_{i_{\#\text{terms}}}$ with
$i_1 < \dots < i_{\#\text{terms}}$, so order does matter.)

There are a total of 2^{k-1}
strong compositions of k, as

we can see by placing either spaces or + signs between a bunch of 1's:

$$\overbrace{1 \quad 1 \quad 1 + 1 + 1 \quad 1 \quad 1 + 1 \quad 1}^{\text{The composition } 3+1+4+2 \text{ of } 9}$$

$$\overbrace{1 + 1 + 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1}^{\text{The composition } 1+1+7 \text{ of } 9}$$

Therefore count the multinomial terms:

$$n^{-k/2}\overbrace{\sum_{\alpha}}^{2^{k-1}\text{ terms}} \overbrace{\sum_{1\lesssim i_1<\cdots<i_{\#\text{terms}}\lesssim n}}^{\binom{n}{\#\text{terms}}\text{ terms}} \overbrace{\binom{k}{\alpha_1,\ldots,\alpha_{\#\text{terms}}}}^{\text{Bounded in } n\ldots\text{ never} > k!} \prod_{i=1}^{\#\text{terms}} \mathbb{E}(X_i^{\alpha_i})$$

$$\alpha_1+\cdots+\alpha_{\#\text{terms}}=k$$

$$(\alpha_i>0)$$

The only part except the $n^{-k/2}$ depending on n is the $\binom{n}{\#\text{terms}}$, which is $O(n^{\#\text{terms}})$. The $n^{-k/2}$ in front will kill the parts corresponding to α with $\#\text{terms} < k/2$, because then

$$O(n^{\#\text{terms}}) = o(n^{k/2}).$$

<u>Conclusion</u>: the only types of terms in the expansion of

$$\mathbb{E}\left[\left(\frac{X_1 + \cdots + X_n}{\sqrt{n}}\right)^k\right]$$

that are plentiful enough not to get killed are those where at least $k/2$ distinct X_i show up.

Now...

At least $k/2$ distinct X_i show up

AND

Any X_i that shows up must have exponent $\neq 1$ (or else that term gets killed by $\mathbb{E}(X_i)=0$)

AND

Sum of exponents is k

$\Longrightarrow$

Only $\mathbb{E}(X_i^2)$ shows up.